Accredited

OCR
Oxford Cambridge and RSA

AS Level Further Mathematics A
Y531 Pure Core
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 15 minutes

MODEL ANSWERS

OCR supplied materials:
- Printed Answer Booklet
- Formulae AS Level Further Mathematics A

You must have:
- Printed Answer Booklet
- Formulae AS Level Further Mathematics A
- Scientific or graphical calculator

INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.**
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do **not** write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\text{m}\,\text{s}^{-2}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

1 **In this question you must show detailed reasoning.**

The equation $x^2 + 2x + 5 = 0$ has roots α and β. The equation $x^2 + px + q = 0$ has roots α^2 and β^2.

Find the values of p and q. **[3]**

$x^2+2x+5=0$

$\hookrightarrow \alpha+\beta=-2 \quad \alpha\beta=5$

for $x^2+px+q=0$:

$\alpha^2+\beta^2=-p$

$(\alpha+\beta)^2-2\alpha\beta=-p$

$-p=(-2)^2-2(5)$

$=4-10$

$\Rightarrow p=6$

$\alpha^2\beta^2=q$

$(\alpha\beta)^2=q$

$q=5^2$

$\Rightarrow q=25$

2 **In this question you must show detailed reasoning.**

Given that $z_1 = 3+2\mathrm{i}$ and $z_2 = -1-\mathrm{i}$, find the following, giving each in the form $a+b\mathrm{i}$.

(i) $z_1^* z_2$ **[2]**

(ii) $\dfrac{z_1+2z_2}{z_2}$ **[2]**

i. $z_1^* z_2 = (3-2i)(-1-i)$

$= -3+2i-3i+2i^2$

$= -5-i$

ii. $\dfrac{z_1+2z_2}{z_2} = \dfrac{3+2i-2-2i}{-1-i}$

$= \dfrac{1}{-1-i}$

$= \dfrac{1}{-1-i} \times \dfrac{-1+i}{-1+i}$

multiply top & bottom by conjugate of denominator

$= \dfrac{-1+i}{1-i^2}$

$= \dfrac{-1+i}{2}$

3 **(i)** You are given two matrices, **A** and **B**, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Show that $\mathbf{AB} = m\mathbf{I}$, where m is a constant to be determined. **[2]**

(ii) You are given two matrices, **C** and **D**, where

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 3 \\ -1 & 2 & 2 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -4 & 8 & -2 \\ -5 & 9 & -1 \\ 3 & -5 & 1 \end{pmatrix}.$$

Show that $\mathbf{C}^{-1} = k\mathbf{D}$ where k is a constant to be determined. **[2]**

i. $$AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1+4 & 2-2 \\ -2+2 & 4-1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow m = 3$$

ii. $$CD = \begin{pmatrix} 2 & 1 & 5 \\ 1 & 1 & 3 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} -4 & 8 & -2 \\ -5 & 9 & -1 \\ 3 & -5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\mathbb{I} \Rightarrow CD = 2\mathbb{I}$$

$$C^{-1}CD = C^{-1}2\mathbb{I}$$

$$D = 2C^{-1}$$

$$C^{-1} = \tfrac{1}{2}D \quad \therefore k = \tfrac{1}{2}$$

(iii) The matrices **E** and **F** are given by $\mathbf{E} = \begin{pmatrix} k & k^2 \\ 3 & 0 \end{pmatrix}$ and $\mathbf{F} = \begin{pmatrix} 2 \\ k \end{pmatrix}$ where k is a constant.

Determine any matrix **F** for which $\mathbf{EF} = \begin{pmatrix} -2k \\ 6 \end{pmatrix}$. **[5]**

iii. $EF = \begin{pmatrix} -2k \\ 6 \end{pmatrix}$

$$\begin{pmatrix} k & k^2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ k \end{pmatrix} = \begin{pmatrix} -2k \\ 6 \end{pmatrix}$$

$\Rightarrow 2k + k^3 = -2k$

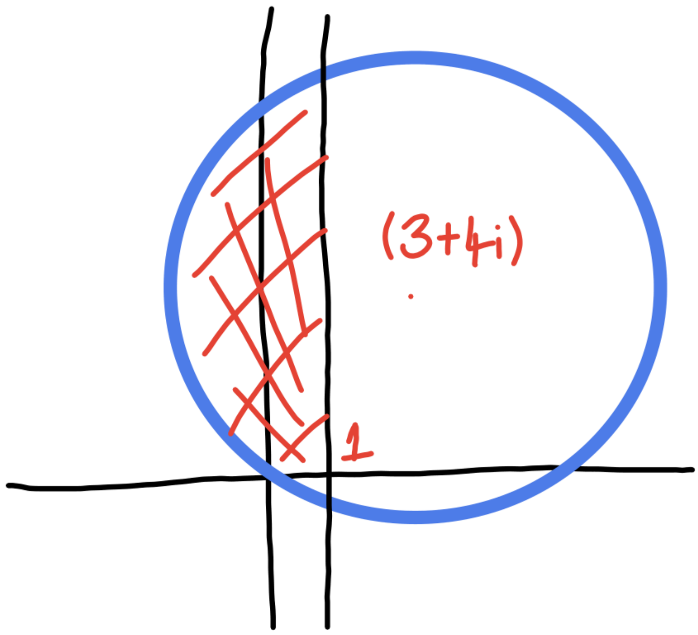
$k^3 + 4k = 0$

$k(k^2+4) = 0$

$\therefore k = 0$ or $k = \pm 2i$

$\underline{F} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2i \end{pmatrix}$ or $\begin{pmatrix} 2 \\ -2i \end{pmatrix}$

4 Draw the region of the Argand diagram for which $|z-3-4\text{i}| \leq 5$ and $|z| \leq |z-2|$. **[4]**



$\sqrt{3^2+4^2} = \sqrt{5^2}$

$\Rightarrow$ circle through origin

5 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

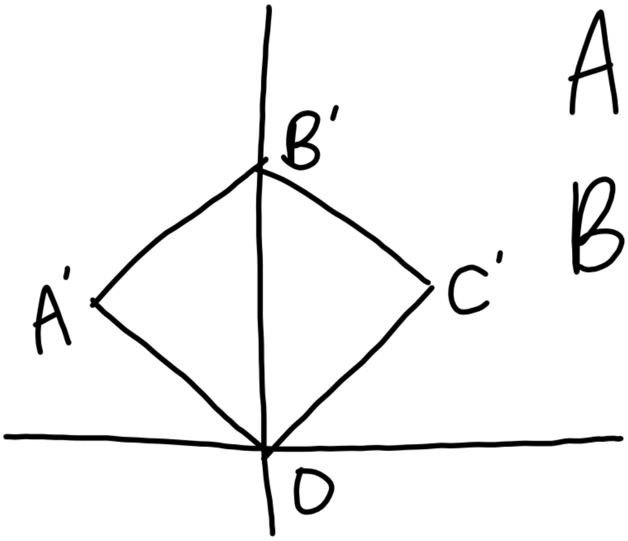
(i) The diagram in the Printed Answer Booklet shows the unit square $OABC$. The image of the unit square under the transformation represented by **M** is $OA'B'C'$. Draw and clearly label $OA'B'C'$. **[3]**

(ii) Find the equation of the line of invariant points of this transformation. **[3]**

(iii) (a) Find the determinant of **M**. **[1]**

(b) Describe briefly how this value relates to the transformation represented by **M**. **[2]**

i.



$A': \left(-\frac{3}{5}, \frac{4}{5}\right)$

$B': \left(\frac{1}{5}, \frac{7}{5}\right)$

$C'\left(\frac{4}{5}, \frac{3}{5}\right)$

ii. invariant points remain the same under transformation

$\Rightarrow M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$-\frac{3}{5}x + \frac{4}{5}y = x$

$\frac{4}{5}y = \frac{8}{5}x$

$\underline{y = 2x}$

$\frac{4}{5}x + \frac{3}{5}y = y$

$\frac{4}{5}x = \frac{2}{5}y$

$\underline{2x = y}$

hence all points on $y=2x$ are invariant

iii. a) det $M = \left(-\frac{3}{5} \times \frac{3}{5}\right) - \left(\frac{4}{5} \times \frac{4}{5}\right) = \frac{-9}{25} - \frac{16}{25}$

$= -1$

b) the area remains the same but the orientation of the image has changed

6 At the beginning of the year John had a total of £2000 in three different accounts. He has twice as much money in the current account as in the savings account.

- The current account has an interest rate of 2.5% per annum.
- The savings account has an interest rate of 3.7% per annum.
- The supersaver account has an interest rate of 4.9% per annum.

John has predicted that he will earn a total interest of £92 by the end of the year.

(i) Model this situation as a matrix equation. **[2]**

(ii) Find the amount that John had in each account at the beginning of the year. **[2]**

(iii) In fact, the interest John will receive is £92 **to the nearest pound**. Explain how this affects the calculations. **[2]**

i. let x be the amount invested in the current account
let y be the amount invested in the savings account
let z be the amount invested in the supersaver account

$$\begin{pmatrix} 1 & 1 & 1 \\ 0.025 & 0.037 & 0.049 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2000 \\ 92 \\ 0 \end{pmatrix}$$

ii. $x + y + z$ ① Solve the simultaneous equations

$0.025x + 0.037y + 0.049z = 92$ ②

$x - 2y = 0$

$x = 2y$ ③

③ in ①: $2y + y + z = 2000$

$3y + z = 2000$

$z = 2000 - 3y$ ④

Sub ③ & ④ in ①

$0.025(2y) + 0.037y + 0.049(2000-3y) = 92$

$0.5y + 0.037y + 98 - 0.147y = 92$

$6 = 0.06y$

$\underline{y = 100}$

$\rightarrow x = 2(100) = 200$

$z = 2000 - 3(100) = 1700$

So invests £100 in savings acc.

£200 in current acc.

£1700 in supersaver acc.

iii. the 92 from (ii) should be 92 ± 0.5, giving a range of answers for each account

7 **In this question you must show detailed reasoning.**

It is given that $f(z) = z^3 - 13z^2 + 65z - 125$.

The points representing the three roots of the equation $f(z) = 0$ are plotted on an Argand diagram.

Show that these points lie on the circle $|z| = k$, where k is a real number to be determined. **[9]**

$f(5) = 0$, hence $(z-5)$ is a factor of $f(z)$

$$\begin{array}{r|l} & z^2 - 8z + 25 \\ \hline (z-5) & z^3 - 13z^2 + 65z - 125 \\ & -(z^3 - 5z^2) \\ \hline & \quad -8z^2 + 65z - 125 \\ & \quad -(-8z^2 + 40z) \\ \hline & \qquad 25z - 125 \\ & \qquad -(25z - 125) \\ \hline & \qquad\qquad 0 \end{array}$$

$$\therefore f(z) = (z-5)(z^2 - 8z + 25)$$

$z = 5$ or $z^2 - 8z + 25 = 0$

$$z = \frac{8 \pm \sqrt{8^2 - 4(25)}}{2}$$

$$= \frac{8 \pm \sqrt{-36}}{2}$$

$$= \frac{8 \pm 6i}{2}$$

$$= 4 \pm 3i$$

find the moduli of these three roots

$|5| = 5$

$|4+3i| = \sqrt{4^2+3^2} = 5$

$|4-3i| = \sqrt{4^2+3^2} = 5$

the points are all an equal distance of 5 from the origin, meaning they all lie on the circle of radius 5.

$\underline{k=5}$

8 Prove that $n! > 2^n$ for $n \geq 4$. **[5]**

prove by induction

base case: let $n=4$, $4! = 24$

$2^4 = 16$

$24 > 16$ so $4! > 2^4$

$\Rightarrow$ hence true for $n=4$

assume true for $n=k$, so $k! > 2^k$

for $n=k+1$: $(k+1)! = (k+1) \times k! > (k+1) \times 2^k$ $\because$ $k! > 2^k$

since $k+1 > 2$ for $k \geq 4$,

$(k+1) \times 2^k > 2 \times 2^k = 2^{k+1}$

$\therefore$ $(k+1)! > 2^{k+1}$

Statement is true for base case & true for k+1 when assumed true for $n=k$

$\therefore$ true for all $n \geq 4$ by induction

9 **(i)** Find the value of k such that $\begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\3\\k\end{pmatrix}$ are perpendicular. **[2]**

Two lines have equations l_1: $\mathbf{r}=\begin{pmatrix}3\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and l_2: $\mathbf{r}=\begin{pmatrix}6\\5\\2\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$.

(ii) Find the point of intersection of l_1 and l_2. **[4]**

(iii) The vector $\begin{pmatrix}1\\a\\b\end{pmatrix}$ is perpendicular to the lines l_1 and l_2.

Find the values of a and b. **[5]**

END OF QUESTION PAPER

i. $\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\3\\k\end{pmatrix}=0$

$-2+6+k=0$

$\underline{k=-4}$

ii. $\begin{pmatrix}3\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}6\\5\\2\end{pmatrix}+\mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$

$3+\lambda=6+2\mu$ $\qquad$ $7+3\lambda=2-\mu$

$\lambda=3+2\mu$ $\qquad$ $\mu=-5-3\lambda$

$\lambda = 3 + 2(-5-3y)$

$= 3 - 10 - 6\lambda$

$7\lambda = -7$

$\lambda = -1 \Rightarrow \mu = -5-3(-1) = -2$

Check with the z equation

$7 + 3\lambda = 2 - \mu$

$7 - 3 = 2 - -2$

$4 = 4$

hence coordinates are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ ✓

iii. $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix} = \underline{i}(2-4) - \underline{j}(-1-6) + \underline{k}(1+2)$

$= \begin{pmatrix} -2 \\ -7 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} \Rightarrow \lambda = -2$

$a = \frac{7}{2}$

$b = -\frac{3}{2}$ ✓